HOMOTOPY THEORY

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B. S., Kansas State College of Pittsburg, 1962

A MASTER'S REPORT

submitted in partial fulfillment of the

requirements for the degree

MASTER OF SCIENCES

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1964

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ACKNOWLEDGEMENTS

I wish to express my thanks to Professor Marr for all his suggestions, helpful discussions and tireless guidance.

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TABLE OF CONTENTS

HOMOTOPY THEORY

INTRODUCTION

The term "Homotopy" was first introduced by Dehn-Heegaard [1,p.39]. The theory of homotopy is an excellent vehicle for introducing a new train of ideas concerning the fundamental group and algebraic topology.

The approach to algebraic topology via topology is found in the strongly geometric flavor of the theory of homotopic mappings. Roughly speaking, two mappings are homotopic in a given space if one can be "deformed" continuously into the other.

The so-called fundamental group is a concept introduced by the French mathematician Jules Henri Poincaré (1854-1912). Therefore the fundamental group is frequently referred to as the Poincaré group. This group plays an important role in the theory of functions. This fundamental group is also referred to as the first homotopy group.

Consider, for example, the case of simple arcs or paths in a topological space Y. By a path is meant the image of a continuous mapping $f: I \rightarrow Y$ where I is the space of real numbers t such that $0 \leq t \leq 1$, together with the usual topology determined by the metric $d(x_1,x_2) = |x_1 - x_2|$.

If α and β are two simple arcs with common initial and terminal points, then α is said to be homotopic to β if it can be "deformed" continuously into β. As the Figure 1 below indicates, there may be a set of arcs beginning at x_0 and

ending at y_0 in a topological space. Some of these arcs are indicated by the dotted lines which indicate the various stages of the deformation. Intuitively, an arc varies continuously from one position to another since deformation is continuous.

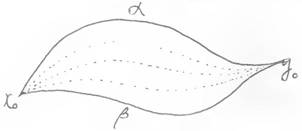

Figure 1

In this manner, one may arrive at the mathematical definition of homotopy of simple arcs with common end points.

HOMOTOPIC RELATIONS AND PROPERTIES

Definition 1: Let X and Y be any two topological spaces, and let f and g be two continuous mappings of X into Y. Then f and g are said to be homotopic if there is a continuous mapping h of X x I into Y such that $h(x,0)=f(x)$, and $h(x,1)=g(x)$ for all $x \in X$ where I is the closed unit interval and X x I denotes the Cartisian product of X and I.

Whenever one has two continuous mappings f and g of a space X into a space Y which are homotopic, the symbol "$f \simeq g$" denotes that these continuous mappings are homotopic.

If $A \subset X$, $f: A \to Y$, and $F: X \to Y$ such that for each $x \in A$, $f(x) = F(x)$, then F is called an extension of f to X and f is called the restriction of F to A. The restriction of F to A

may be denoted by F/A (see 2, p.29).

One could state an equivalent definition. That is to say, two mappings f and g of a space X into a space Y are homotopic if there exists a continuous mapping $h\colon X \times I \to Y$ such that $h/X \times \{0\} = f$ and $h/X \times \{1\} = g$. The mapping h is called a homotopy between f and g, and one calls the product space $X \times I$ the homotopy cylinder.

<u>Definition 2</u>: Let Y be a topological space. A continuous mapping $f\colon [0,1] \to Y$ is called a path in Y. The path f is said to connect or join the point f(0) to the point f(1). The point f(0) is called the initial point, the point f(1) is called the terminal point of the path f, and $f[0,1]$ is called the curve in Y.

Example: Let A be an annulus, that is, A is the collection of points in the plane satisfying $a^2 \le x^2 + y^2 \le b^2$ with $0 < a < b$.

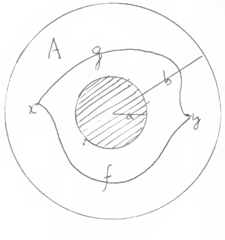

Figure 2

Let f and g be two paths as indicated in Figure 2 joining two fixed points x and y in A. The paths f and g cannot be deformed into each other because of the "hole" determined by $x^2 + y^2 < a^2$. The shaded part denotes the hole.

From the above example one intuitively sees that the paths f and g joining the fixed points x and y cannot be deformed into each other continuously without using points not in the annulus. One needs to introduce the meaning of arcwise connectedness of spaces before further descriptions about homotopy.

Definition 3: A topological space Y is said to be arcwise connected if, for each pair of points y_1, $y_2 \in Y$, there is an arc f contained in Y connecting y_1 to y_2.

Since the meaning of an arcwise connected topological space has been introduced, the following definition concerning homotopy may be given.

Definition 4: Let f and g be two continuous mappings of the unit interval I into an arcwise connected space Y such that $f(0) = g(0) = x$ and $f(1) = g(1) = y$. Then f and g are said to be homotopic with the fixed end points x and y if there exists a continuous mapping

$h: I \times I \to Y$ such that

(1) $h(s,0) = f(s)$,

$h(s,1) = g(s)$ for all $s \in I$,

(2) $h(0,t) = x$,

$h(1,t) = y$ for all $t \in I$.

In particular, if the point x coincides with the point y, then one is led to the following definition.

Definition 5: Let f and g be two paths in a topological space Y both beginning and ending at the same point y of Y. Then f and g are said to be homotopic with respect to the fixed point y if there exists a continuous mapping $h: I \times I \rightarrow Y$ such that

$$h(s,0) = f(s),$$
$$h(s,1) = g(s) \text{ for all } s \in I,$$
$$h(0,t) = h(1,t) = y \text{ for all } t \in I.$$

In the above definition, one has a closed curve defined on a topological space Y with respect to the fixed point y. Intuitively, if it is possible to shrink the curve continuously to the point y in the space Y, one arrives at the following definition.

Definition 6: A closed path f in a topological space Y starting and ending at a fixed point y will be said to be shrinkable to y, or to be homotopic to a constant with respect to the base-point y, if f is homotopic to the constant map $e: I \rightarrow Y$ defined by $e(s) = y$ for all $s \in I$.

Diagramatically, one may visualize the mapping $h: I \times I \rightarrow Y$ being constructed in such a way that just restricting the lower side $t = 0$ in a unit square as indicative in Figure 3 below, one would have the path $f(s) = h(s,0)$ alone while the other three sides of the unit square I^2 are mapped into the fixed point y.

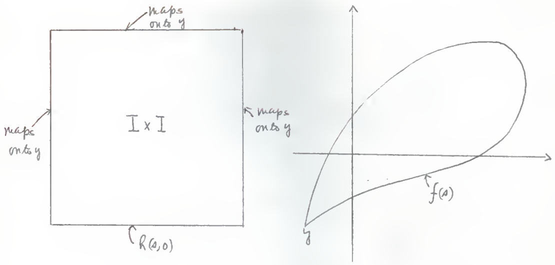

Figure 3

One might ask as to what happens if lines are drawn parallel to $t = 0$ in the unit square, and what would their mappings be like then? The following Figure 4 will indicate the specific mappings.

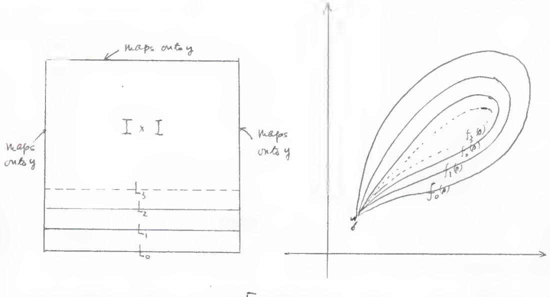

Figure 4

The figures indicate the mappings $f_o(s)$, $f_1(s)$, $f_2(s)$, and

$f_3(s)$ corresponding to the horizontal lines L_0, L_1, L_2, and L_3 respectively. It is intuitively clear that the mappings $f_i (i = 0,1,2,3)$ are shrinking with respect to the fixed point y in the topological space Y. Since the basic notions about homotopy with respect to a base-point y have been introduced, the idea of the equivalence classes is introduced. The most significant feature of an equivalence relation is that it divides a set into mutually exclusive subsets. These subsets are called equivalence classes.

Definition 7: Let R be an equivalence relation defined on a set S, and let $a \in S$. The subset S consisting of all elements x S such that aRx is called the equivalence class of a and is denoted by $[a]$.

Theorem 1: Let R be an equivalence relation defined on a set S; then the equivalence classes constitute a decomposition of S into disjoint subsets.

Proof: For every a S, it is true that $a \in [a]$. Thus, the equivalence class is non-empty, and S is the union of all these equivalence classes.

Now, it remains to show that two equivalence classes are either disjoint or identical. Assume $[a] \cap [b] \neq \phi$. Suppose $x \in [a] \cap [b]$. This implies that $x \in [a]$and $x \in [b]$. Suppose $m \in [a]$; then aRm holds. Then mRa and aRx implies mRx holds. Since xRb and mRx hold, then mRb holds and $m \in [b]$. Hence $[a] \subset [b]$. Similarly it may be shown that $[b] \subset [a]$. Then $[a] \subset [b]$ and $[b] \subset [a]$ implies $[a] = [b]$.

<u>Theorem 2</u>: The homotopy relation between mappings of a space I into a space Y is an equivalence relation on Y^I, where the function space [3, p.30] Y^I is the collection of all continuous mappings of a space I into a space Y.

Proof: To prove that the homotopy relation is an equivalence relation, it suffices to show the following:

(1) $f \simeq f$ for every $f \in Y^I$ (reflexive law)

(2) $f \simeq g$ implies $g \simeq f$ where $f, g \in Y^I$ (symmetric law)

(3) $f \simeq g$ and $g \simeq k$ implies $f \simeq k$ for f, g, $k \in Y^I$ (transitive law).

Let f be any mapping in the function space Y^I. Define $h: X \times I \rightarrow Y$ by $h(x,t) = f(x)$, $(0 \leq t \leq 1)$. Since $f \in Y^I$ where Y^I is a collection of all continuous mappings, it is obvious that h is continuous, and that $h(x,0) = f(x) = h(x,1)$ for all $x \in X$. This establishes the reflexive law.

If $f \simeq g$, by definition, there is a homotopy $h: X \times I \rightarrow Y$ such that $h(x,0) = f(x)$, and $h(x,1) = g(x)$ for all $x \in X$.

Define $h'(x,t) = h(x,1-t)$. Then, $h'(x,t)$ is continuous, and $h'(x,0) = h(x,1-0) = h(x,1) = g(x)$, and $h'(x,1) = h(x,1-1)$ $h(x,0) = f(x)$. Thus $g \simeq f$. This establishes the symmetric law.

If $f \simeq g$ and $g \simeq k$ where f, g, $k \in Y^I$, then there are homotopies h_1 and h_2, with

$$h_1(x,0) = f(x), \qquad h_1(x,1) = g(x)$$
$$h_2(x,0) = g(x), \qquad h_2(x,1) = k(x).$$

Define a homotopy h between f and k by setting

$$h(x,t) = h_1(x,2t), \qquad (0 \leq t \leq \tfrac{1}{2})$$
$$h(x,t) = h_2(x,2t-1), \qquad (\tfrac{1}{2} \leq t \leq 1).$$

The intervals of t for h_1 and h_2 are closed. When $t = \frac{1}{2}$

$$h(x,\tfrac{1}{2}) = h_1(x,\ 2 \times (\tfrac{1}{2})) = h_1(x,1) = g(x), \text{ while}$$

$$h(x,\tfrac{1}{2}) = h_2(x,\ 2 \times (\tfrac{1}{2}-1)) - h_2(x,0) = g(x).$$

Therefore h is well-defined and is continuous on X x I.

Moreover,

$$h(x,0) = h_1(x,0) = f(x),$$

$$h(x,1) = h_2(x,\ 2-1) \quad = h_2(x,1) = k(x).$$

Therefore f k. The proof is complete.

Definition 8: The equivalence classes of paths based on y Y corresponding to the relation of homotopy with respect to the base-point y will be called the homotopy class of paths on Y with respect to the base-point y.

In studying the theory of homotopy, one may come across a case where the curve traced out by f does enclose some "hole" or "holes" in the space Y. In order to clarify this notion the property of a space being simply connected must be considered.

Definition 9: A topological space Y is said to be simply connected if at each point $y \in Y$ there is only one homotopy class of closed paths.

It is obvious that for simply connected spaces, any closed path at y is homotopic to the constant path at y.

Definition 10: A topological space Y is said to be locally connected at a point $y \in Y$ if each neighborhood N of y contains a connected neighborhood M of y. A topological space Y is said to be locally connected if it is locally connected at each of its points.

The Italian mathematician and logician Giuseppe Peano (1858-1932) presented in 1890 a space-filling curve. Many constructions of such "Peano curves" have appeared since.

Definition 11: A Peano space or a Peano continuum is a compact, connected, and local connected metric space.

Definition 12: A subset C of a space S is called a component of S provided that C is connected but is not a proper subset of another connected set in the same space S.

Theorem 3: The homotopy classes of Y^I are precisely the arcwise-connected components of Y^I.

Proof: The proof is quite simple. If $f \simeq g$, then the homotopy $h(x,t)$ between f and g yields a mapping $G: I \rightarrow Y^I$ by $G(t) = f_t(x) = h(x,t)$. Then $G(I)$ is a Peano continuum. Thus, it contains an arc between f and g.

Conversely, the arc of the mappings between f and g yields a homotopy $G(t) = f_t(x) = h(x,t)$ between the two continuous mappings f and g.

The following theorem is a very interesting and importnat one discovered by Borsuk, the elegant proof being done by Dowker. This theorem associates homotopy and the extension of mappings.

Theorem 4: Let A be a closed subset of a separable metric space M. Let f' and g' be homotopic mappings of A into the n-sphere S^n. Suppose there exists an extension f of f' to all of M. Then there will also exist an extension g of g' to all of M. The extension f and g may be so chosen that f and g

are homotopic.

Proof: Let $h': A \times I \to S^n$ be a given homotopy between f' and g', and let f be the given extension of f' to all of M. Let E be the set in the Cartisian product $M \times I$ given by $E(A \times I) = (M \times 0)$. Clearly E is a closed subset of $M \times I$.

Now define the mapping $F: E \to S^n$ given by E such that

$$F(x,0) = f(x) \qquad \forall x \in M$$

$$F(x,t) = h'(x,t) \qquad \forall x \in A, \text{ and } 0 \leq t \leq 1.$$

Since $h'(x,0) = f'(x) = f(x)$, $\forall x \in A$, the mapping F is well-defined and continuous.

A theorem in topology (4, p.63) states that there is an open set X in $M \times I$ such that X contains E and such that F can be extended to a mapping F' on X. There is an open set Y in M such that Y contains A and such that $Y \times I$ lies in X. Thus the mapping F is defined on $(Y \times I) \cup (M \times 0)$.

Since Y contains A, therefore, F can be applied to give a mapping $G(x,t)$ which agrees with F' on the set $(A \times I) \cup (M \times 0)$. Let $g(x)$ be $G(x,1)$; then it will give the desired extension of g'.

HOMOTOPY PATHS

Before investigating the ideas connected with the fundamental group, one needs to pave the road for a new train of ideas about the Poincaré group by introducing more background concerning paths.

<u>Definition 13</u>: If $\alpha : I \to Y$ and $\beta : I \to Y$ are two paths, and if $\alpha(1) = \beta(0)$, so that the end of α coincides with the beginning of β, then the product path $\alpha\beta$ of the paths is

defined to be the path $\gamma : I \to Y$ where

$$\gamma(x) = \alpha(2x); \qquad 0 \le x \le \tfrac{1}{2}$$
$$\gamma(x) = \beta(2x-1); \qquad \tfrac{1}{2} \le x \le 1.$$

Intuitively, $\alpha\beta$ can be thought of as the path α followed by the path β. The path γ defined in this manner is continuous.

Theorem 5: If a topological space S is the union of two closed subsets A and B, and if $f: A \to T$ and $g: B \to T$ are continuous mappings of A and B respectively into a space T such that $f(x) = g(x)$, $\forall x \in A \cap B$, then the transformation h: S T defined by $h(x) = f(x)$ if $x \in A$, and $h(x) = g(x)$ if $x \in B$, is continuous.

Proof: Let $X \subset S$. Define $X_1 = S \cap A$ and $X_2 = X \cap B$. Then $X = X_1 \cup X_2$, and so $h(X) = h(X_1) \cup h(X_2)$.

Since $\overline{X_1 \cup X_2} = \overline{X}_1 \cup \overline{X}_2$ and $\overline{X} = \overline{X}_1 \cup \overline{X}_2$, then $h(\overline{X}) = h(\overline{X_1}) \cup h(\overline{X_2})$. Since $X_1 = X \cap A$, then $\overline{X_1} = \overline{X \cap A}$.

Let $x \in \overline{X}_1 = \overline{X \cap A}$. Then every open set containing x contains a point of $X \cap A$, and therefore of A, and hence $x \in \overline{A}$. Since A is closed, $x \in A$. Since $\overline{X}_1 \subset A$, $h(\overline{X}_1) = f(\overline{X_1})$. Similarly $\overline{X_2} \subset B$, so that $h(\overline{X}_2) = g(\overline{X}_2)$. A map $k: T_1 \to T_2$ is continuous if and only if for every subset $X \subset T_1$, $k(\overline{X}) \subset \overline{k(X)}$.

Now, $f(\overline{X_1}) \subset \overline{f(X_1)}$ and $g(\overline{X_2}) \subset \overline{g(X_2)}$. Therefore, $h(\overline{X}) = h(\overline{X_1}) \cup h(\overline{X_2}) \subset \overline{f(X_1)} \cup \overline{g(X_2)} = \overline{h(X_1)} \cup \overline{h(X_2)}$, and $\overline{h(X_1)} \cup \overline{h(X_2)} = \overline{h(X_1) \cup h(X_2)} = \overline{h(X)}$. Therefore $h(\overline{X}) \subset \overline{h(X)}$ and h is continuous.

Definition 14: A path $\alpha : I \to T$ for which $\alpha(I)$ is a single element of T is called a null path.

As it has been mentioned before, a path α is said to be closed if its end coincides with its beginning; i.e., if

$\alpha(0) = \alpha(1)$, then α is a closed path.

Now, let one be reminded of the meaning of the inverse path again before the next theorem is introduced. The inverse path f^{-1} of a given path is defined to be the path given by $f^{-1}(t) = f(1-t)$, where $0 \leq t \leq 1$. For example, $\alpha\alpha^{-1}$ and $\alpha^{-1}\alpha$ are closed paths.

<u>Theorem 6</u>: If f, g, f', g' are paths in Y such that $f \simeq f'$ and $g \simeq g'$, and if fg exists, then f'g' exists and fg f'g'.

Proof: First of all, let the existence of f'g' be shown. Since fg is defined, $f(1) = g(0)$. Since $f \simeq f'$ and $g \simeq g'$, $f(1) = f'(1)$ and $g(0) = g'(0)$. Therefore $f'(1) = g'(0)$ and f'g' exists.

Since $f \simeq f'$, there exists a continuous mapping $F\colon I \times I \rightarrow Y$ such that $F(x,0) = f(x)$, $F(x,1) = f'(x)$, $F(0,t) = f(0) = f'(0)$, and $F(1,t) = f(1) = f'(1)$. Similarly, since $g \simeq g'$, there exists a continuous mapping $G\colon I \times I \rightarrow Y$ such that $G(x,0) = g(x)$, $G(x,1) = g'(x)$, $G(0,t) = g(0) = g'(0)$, and $G(1,t) = g(1) = g'(1)$.

Now define H: I x I Y by

$$H(x,t) = F(2x,t), \qquad (0 \leq x \leq \tfrac{1}{2})$$
$$H(x,t) = G(2x-1,t), \qquad (\tfrac{1}{2} \leq x \leq 1).$$

By Theorem 5, H is continuous, and so

$$H(x,0) = f(2x), \qquad (0 \leq x \leq \tfrac{1}{2})$$
$$H(x,0) = g(2x-1), \qquad (\tfrac{1}{2} \leq x \leq 1)$$
$$H(x,1) = f'(2x), \qquad (0 \leq x \leq \tfrac{1}{2})$$
$$H(x,1) = g'(2x-1), \qquad (\tfrac{1}{2} \leq x \leq 1)$$
$$H(0,t) = F(0,t) = f(0) = f'(0)$$

$$E(1,t) = G(1,t) = g(1) = g'(1).$$

Therefore we conclude $fg \simeq f'g'$.

Theorem 7: If f and g are closed paths such that $f \simeq g$, then $f^{-1} \simeq g^{-1}$.

Proof: Since $f \simeq g$, there is a continuous mapping $F\colon I \times I \to Y$ such that $F(x,0) = f(x)$, $F(x,1) = g(x)$, $F(0,t) = f(0) = g(0)$, and $F(1,t) = f(1) = g(1)$.

Define $G\colon I \times I \to Y$ by $G(x,t) = F(1-x,t)$. Then G is continuous and $G(x,0) = f(1-x)$, $G(x,1) = g(1-x)$, $G(0,t) = F(1,t) = f(1) = g(1)$, and $G(1,t) = F(0,t) = f(0) = g(0)$. Therefore $f^{-1} \simeq g^{-1}$.

Theorem 8: If f is any path and g is a null path such that fg exists, then $fg \simeq f$. Similarly, if h is a null path such that hf exists, then $hf \approx f$.

Proof: Since fg exists and g is a null path which exists, $g(x) = f(1)$ for $0 \leq x \leq 1$.

Define $F\colon I \times I \to Y$ by

$$F(x,t) = f\left(\frac{2x}{1+t}\right), \qquad (0 \leq x \leq \tfrac{1}{2}(1+t))$$

$$F(x,t) = f(1), \qquad (\tfrac{1}{2}(1+t) \leq x \leq 1)$$

Then

$$F(x,0) = f(2x) \text{ for } (0 \leq x \leq \tfrac{1}{2}),$$

$$F(x,0) = f(1) \text{ for } (\tfrac{1}{2} \leq x \leq 1),$$

$$F(0,t) = f(0),$$

$$F(1,t) = f(1).$$

Also F is continuous. Therefore $fg \simeq f$. Similarly $hf \simeq f$.

Theorem 9: If f, g and h are three paths such that fg and gh exist, then $(fg)h$ and $f(gh)$ both exist, and $(fg)h \simeq f(gh)$.

Proof: Let the mapping $F: I \times I \to Y$ be defined by

$$F(x,t) = f\left(\frac{4x}{1+t}\right), \text{ where } 0 \le x \le \tfrac{1}{4}(1+t),$$

$$F(x,t) = g(4x-1-t), \text{ where } \tfrac{1}{4}(1+t) \le x \le \tfrac{1}{4}(2+t),$$

$$F(x,t) = h\left(1-\frac{4(1-x)}{2-t}\right) \text{ where } \tfrac{1}{4}(2+t) \le x \le 1.$$

The mapping is continuous by Theorem 5. Then

$$F(x,0) = f(4x) \text{ for } 0 \le x \le \tfrac{1}{4},$$

$$F(x,0) = g(4x-1) \text{ for } \tfrac{1}{4} \le x \le \tfrac{1}{2},$$

$$F(x,0) = h(2x-1) \text{ for } \tfrac{1}{2} \le x \le 1,$$

$$F(x,1) = f(2x) \text{ for } 0 \le x \le \tfrac{1}{2},$$

$$F(x,1) = g(4x-2) \text{ for } \tfrac{1}{2} \le x \le 3/4,$$

$$F(x,1) = h(4x-3) \text{ for } 3/4 \le x \le 1.$$

Hence $F(x,0) = k_0(x)$ where $(fg)h = k_0$, and

$$F(x,1) = k_1(x) \text{ where } f(gh) = k_1. \text{ Also}$$

$$F(0,t) = f(0) = k_0(0),$$

$$F(1,t) = h(1) = k_1(1).$$

Therefore $(fg)h \simeq f(gh)$.

<u>Theorem 10</u>: If f is any path, then ff^{-1} and $f^{-1}f$ are homotopic to null paths.

Proof: Define $F: I \times I \to Y$ by

$$F(x,t) = f[(2x)(1-t)], \qquad 0 \le x \le \tfrac{1}{2}$$

$$F(x,t) = f(2(1-x)(1-t)), \qquad \tfrac{1}{2} \le x \le 1.$$

Then F is continuous, and

$$F(x,0) = f(2x), \qquad 0 \le x \le \tfrac{1}{2}$$

$$F(x,0) = f(2-2x), \qquad \tfrac{1}{2} \le x \le 1$$

$$F(x,1) = f(0),$$

$$F(0,t) = f(0) = F(1,t).$$

Therefore ff^{-1} is homotopic to the null path whose image is $f(0)$. Similarly, $f^{-1}f$ is homotopic to the null path whose image is $f(1)$.

Theorem 11: Let f and g be two paths such that fg^{-1} exists and is a closed path. Then fg^{-1} is homotopic to a null path if and only if $f \simeq g$.

Proof: Since fg^{-1} exists, $f(1) = g(1)$ because fg^{-1} can be thought of as the path f followed by the path g^{-1}, where $g^{-1}(x)$ is defined as $g(1-x)$. Since fg^{-1} is closed, $f(0) = g(0)$.

Suppose that fg^{-1} is homotopic to a null path. Then by Theorem 6 and Theorem 8, $(fg^{-1})g$ is homotopic to g. By Theorem 9, $(fg^{-1})g$ is homotopic to $f(g^{-1}g)$, which, by Theorem 8 and Theorem 10, is homotopic to f. Hence $f \simeq g$ by the law of transitivity. Conversely, suppose that $f \simeq g$. Then by Theorem 6, $fg^{-1} \simeq gg^{-1}$, and by Theorem 10 gg^{-1} is homotopic to a null path. Hence fg^{-1} is homotopic to a null path by the law of transitivity.

THE FUNDAMENTAL GROUP

Let Y be a topological space, and y_0 be a fixed point of Y. Consider the collection of all continuous mappings $f: I \to Y$ of the unit interval into Y such that $f(0) = y_0 = f(1)$, and denote this collection by $C(Y,y_0)$. In other words, $C(Y,y_0)$ is the set of all closed paths which begin and end at y_0. The point y_0 is called the base point for the paths, and the paths are referred to as "paths at y_0".

This $C(Y,y_0)$ is a subspace of the functional space Y^I.

If f is a path in $C(Y,y_o)$, let $[f]$ denote the homotopy class of which the path f in $C(Y,y_o)$ is a representative. That is, $[f]$ denotes the collection of all g of $C(Y,y_o)$ such that $f \simeq g$ at y_o.

Multiplication of these classes of continuous mappings is defined by the rule $[f] \circ [g] = [f * g]$ where $f * g$ is defined by $(f * g)(x) = f(2x)$ for $0 \leq x \leq \frac{1}{2}$ and $(f * g)(x) = g(2x-1)$ for $\frac{1}{2} \leq x \leq 1$. The collection of all $[f]$ make up a set called $\pi_1(Y,y_o)$. The collection $\pi_1(Y,y_o)$ will become the so-called fundamental group of Y modulo y_o where a group operation is defined on $[f]$ in $\pi_1(Y,y_o)$. This fundamental group is also called the Poincaré group or equivalently, the first homotopy group of Y modulo y_o.

By Theorem 6, this definition $[f] \circ [g] = [f * g]$ is independent of the choice of representatives of $[f]$ and $[g]$, for if $f \simeq f'$ and $g \simeq g'$, then $fg \simeq f'g'$. Hence $[f'] \circ [g'] = [f' * g'] = [f * g]$. Thus, the product $[f] \circ [g]$ is uniquely defined by $[f]$ and $[g]$.

By means of the Theorem 6 to 11 in the foregoing context, one can show that the operation "$\circ$" defines a group structure in the set of homotopy classes of continuous mappings at y_o where the domain of the mappings is the unit interval.

It is quite straightforward to verify the four conditions for a group [5, p.126].

(1). The product $[f] \circ [g]$ is a homotopy class of paths at y_o by Definition 13. Thus, the operation is closed.

(2). $([f] \circ [g]) \circ [h] = [f * g] \circ [h] = [(f * g) * h]$

$$[f] \circ ([g] \circ [h]) = [f] \circ [g * h] = [f * (g * h)]$$
$$\forall [f], [g], [h] \in \Pi_1(Y, y_0).$$

By Theorem 9, $(fg)h \quad f(gh)$. Therefore the operation is associative.

(3). Let $[e]$ denote the homotopy class of the null path at y_0. Then by Theorem 8, $[f] \circ [e] = [f] = [e] \circ [f] \quad \forall [f] \in \Pi_1(Y, y_0)$. Thus $[e]$ is the identity element in $\Pi_1(Y, y_0)$ for the operation " $\circ$ ".

(4). By Theorem 10, $[f] \circ [f^{-1}] = [f * f^{-1}] = [e]$, and $[f^{-1}] \circ [f] = [f^{-1} * f] = [e]$. Thus for every element in $\Pi_1(Y, y_0)$ there exists in $\Pi_1(Y, y_0)$ an inverse with respect to the operation " $\circ$ ". Hence, the set of all homotopy classes of paths beginning and ending at y_0 with multiplications defined in this way is a group. This group is the so-called fundamental group at y_0, and is denoted by $\Pi_1(Y, y_0)$.

PROPERTIES OF THE FUNDAMENTAL GROUP

When one compares the groups $\Pi_1(Y, y_1)$ and $\Pi_2(Y, y_2)$ in the case where y_1 and y_2 can be joined by a path in the topological space Y, it is clear that a given path based on y_1 leads by a simple construction to a path based on y_2. The figures below might help to clarify this.

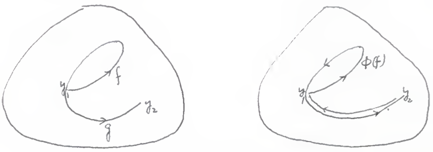

Figure 5

If f is a given path based on y_1 and g is a path from y_1 to y_2, (see Figure 5), then a path $\phi(f)$ based on y_2 is obtained by going along g inversely (that is, from y_2 to y_1), then going around f, and finally back to y_2 along g.

It will now be shown that this correspondence between paths based on y_1 and those based on y_2 lead to an isomorphism between the corresponding groups of homotopy classes.

Definition 15: An isomorphism Φ between two groups G_1, G_2 is a one-one transformation $\Phi: G_1 \rightarrow G_2$ of G_1 onto G_2 which preserves the group operation. That is, $\Phi(g_1 g_1') = \Phi(g_1)\,\Phi(g_1')$ for any pair g_1 and $g_1' \in G_1$.

It is not difficult to show that the fundamental groups at any two points of an arcwise-connected space are isomorphic. In short, one concludes that the fundamental group is essentially independent of the bases point y_0 for such spaces.

Theorem 12: If Y is an arcwise-connected topological space, and if y_0, y_1 are any two points of Y, then $\pi_1(Y, y_0)$ is isomorphic to $\pi_1(Y, y_1)$.

Proof: Let f be a closed path at $y_0 \in Y$. Since Y is arcwise-connected, there exists a path h in Y such that $h(0) = y_0$ and $h(1) = y_1$; that is, y_0 is the beginning point and y_1 is the end point of h. Then $g = (h^{-1}f)h$ is a closed path beginning and ending at y_1. Let a transformation Φ of the homotopy classes of paths at y_0 into the homotopy classes of paths at y_1 be defined by $\Phi[f] = [g]$ where g $(h^{-1}f)h$ while $\Phi[f]$ is uniquely determined by $[f]$.

Conversely, $[g]$ is uniquely determined by Φf, for if $(h^{-1}f_1)h \simeq (h^{-1}f_2)h$, by Theorem 9 and Theorem 10, then $f_1 \simeq f_2$.

If g is any given path at y_1, then g is homotopic to $h^{-1}((hg)h^{-1})h$ and so every homotopy class of paths at y_1 is of the form $\Phi[f]$ for some $[f] \in \pi_1(Y,y_0)$. Thus Φ is a one-to-one transformation of the elements of (Y,y_0) onto the elements of $\pi_1(Y,y_1)$. If $[f_1]$ and $[f_2]$ are two elements of $\pi_1(Y,y_0)$, then $\Phi[f_1]\,\Phi[f_2] = ((h^{-1}f_1)h)((h^{-1}f_2)h) \simeq \Phi[f_1f_2]$. By Theorem 9 and Theorem 10, Φ is an isomorphism.

Definition 16: An arcwise-connected space whose fundamental group is abelian is called 1-simple.

Definition 17: A space X is called a Hopf space if there exists a mapping $\Phi: X \times X \to X$ and a point x_0 of X such that $\Phi(x_0,x_0) = x_0$, and such that both $\Phi(x_0,y): X \to X$ and $\Phi(y,x_0): X \to X$ are homotopic to the identity mapping, the homotopy leaving x_0 fixed.

Theorem 13: An arcwise-connected Hopf space has an abelian fundamental group [4, p.167].

Definition 18: A triple $(G, \circ, \mathcal{J})$ is a topological group [6, p.105] if and only if $(G, \circ)$ is a group, $(G, \mathcal{J})$ is a topological space, and the function whose value at a member (x,y) of $G \times G$ is $x \circ y^{-1}$, is continuous relative to the product topology for $G \times G$.

Corollary 13.1: Every arcwise-connected topological group has an abelian fundamental group.

Proof: If one could show every arcwise-connected topological

group G is a Hopf space then the previous theorem ensures that an arcwise-connected Hopf space has an abelian group. Now, let e be the identity G, and define $\phi : G \times G \to G$ by $\phi(x,y) = x \circ y$. Then obviously $\phi(e,e) = e$, $\phi(e,x) = e \circ x = x$, and $\phi(x,e) = x \circ e = x$, $\forall x \in G$.

Then ϕ is continuous since the group multiplication is continuous, and thus G is a Hopf space.

One might examine the relationship between the fundamental groups of different spaces by considering the effect of a continuous transformation of a given space into another space. A very important concept that one should always keep in mind is that the fundamental group of the image of the transformation need not be isomorphic to the fundamental group of the original space; however, there is a relationship between the structure of the groups. There exists a homomorphism between a group G_1 and a group G_2. A homomorphism between G_1 and G_2 is a many to one transformation $\Phi : G_1 \to G_2$ such that $\Phi(g_1, g_1{}') = \Phi(g_1)\,\Phi(g_1{}')$ for every pair of elements g_1, $g_1{}'$ of G_1.

If A is a closed subset of a space X, then one may speak of the pair of spaces (X,A). By a mapping $f: (X,A) \to (Y,B)$ of the pair (X,A) into the pair (Y,B), is meant a mapping $f: X \to Y$ such that $f(A) \subset B$.

Theorem 14: A mapping $f : (X,x_0) \to (Y,y_0)$ induces a homomorphism $f^* : \pi_1(X,x_0) \to \pi_1(Y,y_0)$.

Proof: Let p, q be closed paths at the point $x_0 \in X$. Define $m,n: I \to Y$ by $m = f \circ p$ and $n = f \circ q$ respectively. Thus

$m(x) = f \circ p(x)$ and $n(x) = f \circ q(x)$. Then m, n are closed paths in Y with base point $y_0 = f(x_0)$.

If $p \simeq q$, then there exists a continuous mapping F: $I \times I \to X$ such that

$$F(x,0) = p(x),$$
$$F(x,1) = q(x),$$
$$F(0,t) = p(0) = q(0) = x_0,$$
$$F(1,t) = p(1) = q(1) = x_0.$$

Define G: $I \times I \to Y$ by $G(x,t) = f(F(x,t)0$. Then G is continuous because the composite mapping theorem states that if f is a mapping of X into Y, and g is a mapping of Y into Z, then the transformation $h = g \circ f$ is a mapping of X into Z.

Moreover,

$$G(x,0) = f \circ p(x) = m(x),$$
$$G(x,1) = f \circ q(x) = n(x),$$
$$G(0,t) = f(x_0) = y_0,$$
$$G(1,t) = f(x_0) = y_0.$$

Therefore, $m \simeq n$.

If one defines $f^*[p] = [f \circ p]$, then f^* is a transformation of homotopy classes of paths in $C(X,x_0)$ into homotopy classes of paths in $C(Y,y_0)$ such that $f^*[p]$ is uniquely determined by $[p]$. Thus, this transformation is said to be induced by f. Hence, with each member of $\pi_1(X,x_0)$, the induced transformation f^* associates a unique member of $\pi_1(Y,y_0)$.

Let p, q be two paths in X at x_0, and let $m = f \circ p$ and $n = f \circ q$ be the corresponding paths in Y. Then $p \circ q$ is the

path γ : I X defined by

$$\gamma(x) = p(2x) \text{ for } 0 \le x \le \tfrac{1}{2},$$

$$\gamma(x) = g(2x-1) \text{ for } \tfrac{1}{2} \le x \le 1.$$

Therefore $f \circ \gamma$ is the path s: $I \to Y$ defined by

$$s(x) = f(p(2x)) \text{ for } 0 \le x \le \tfrac{1}{2},$$

$$s(x) = f(q(2x-1)) \text{ for } \tfrac{1}{2} \le x \le 1,$$

and is in fact the path $m \circ n$. Therefore $f^*[p]\ f^*[q] = [m][n]$ $[mn] = f^*[pq]$. Hence f^* is a homomorphism of $\pi_1(X,x_0)$ into $\pi_1(Y,y_0)$ where $y_0 = f(x_0)$.

One might ask under what condition or conditions the homomorphism of the last theorem would become an isomorphism. The answer is quite simple. If f is a homeomorphism, then f^* is an isomorphism.

<u>Theorem 15</u>: If f and g are homotopic mappings of (X,x_0) into (Y,y_0), then the induced homomorphisms coincide. If f: $(X,x_0) \to (Y,y_0)$ and g: $(Y,y_0) \to (Z,z_0)$, then $(gf)^* = g^* f^*$.

Proof: To prove the induced homomorphisms coincide, it suffices to show the induced homomorphism depends only upon the homotopy class of the mapping. If f and g are homotopic with the fixed point x_0, then the composite mappings $f(p(t))$ and $g(p(t))$ are also homotopic for any $p(t) \in C(X,x_0)$ with the point y_0 fixed.

Therefore f^*p and g^*p are homotopic, and yield $f^* = g^*$. To prove the second part of the theorem, let $p \in C(X,x_0)$. Then

$$[gf^*p]\ (x) = (gf)(p(x)) = g[f(p(x)]$$

$$[gf^*p]\ (x) = g^*[f(p(x))] = g^*[\ f^*(p(x))]$$

$$[gf^* p]\ (x) = [(g^* f^*)p](x)$$

Then it follows immediately that $(gf)^* = g^* f^*$.

Besides, the important properties of the induced homomorphism in the above theorem, the following corollary is worthy of mention. The definition of homotopically equivalent spaces states that two spaces X and Y are of the same homotopy type if there exist mappings $f\colon X \to Y$ and $g\colon Y \to X$ such that the composite mappings $fg\colon Y \to Y$ and $gf\colon X \to X$ are homotopic respectively to the identity mappings $i(y)\colon Y \to Y$ and $i(x)\colon X \to X$.

Corollary 15.1: If (X,x_0) and (Y,y_0) are homotopically equivalent, then $\pi_1(X,x_0)$ is isomorphic to $\pi_1(Y,y_0)$.

Proof: Since (X,x_0) and (Y,y_0) are homotopically equivalent, there exist mappings $f\colon X \to Y$ and $g\colon Y \to X$ such that fg and gf are homotopic to the identity mappings $i(y)\colon Y \to Y$ and $i(x)\colon X \to X$, respectively by definition.

Thus, both $(fg)^* = f^* g^*$ and $(gf)^* = g^* f^*$ are onto isomorphisms by the above theorem. Then f^* is onto because $f^* g^*$ is onto, and since $g^* f^*$ is an isomorphism, it follows that f^* is an isomorphism. Therefore f^* is an isomorphic mapping of $\pi_1(X,x_0)$ onto $\pi_1(Y,y_0)$.

The above corollary leads also to the development of the following corollary.

Corollary 15.2: If X is homeomorphic to Y under f, then $\pi_1(X,x_0)$ is isomorphic with $\pi_1(Y,y_0)$ where y_0 is the image of x_0 under the homeomorphism f.

Proof: Take $f\colon X \to Y$ to be a homeomorphism. Then f^{-1} is

the g in the definition of homotopically equivalent. Then Corollary 15.1 states that $\pi_1(X,x_0)$ is isomorphic to $\pi_1(Y,y_0)$ where $y_0 = f(x_0)$ under homeomorphism.

Before mentioning further results of some important properties of the fundamental group, some examples will be given.

Example I: The fundamental group of the sphere consists of a single element, the identity, because all closed paths are homotopic to a null path.

In general, a space whose fundamental group consists of a single element, the identity alone, is called simply connected. That is, in a simply connected space, every closed curve is homotopic to a point.

Example II: The fundamental group of the circle is an infinite cyclic group. Suppose p is any point on the circle. Then a closed path starting and terminating at p is homotopic to either a null path or to a path given by one or more complete descriptions of the circle. Let α be a path describing the circle "a" times, and let β be another path which describes it "b" times in the same direction. Let $a > b$. Then α is not homotopic to β for $\alpha\beta^{-1}$ is a path describing the circle "a-b" times, and such a path is not homotopic to a null path.

If $\alpha \simeq \beta$, this implies that $\alpha\beta^{-1} \simeq \beta\beta^{-1} = e$. Thus the homotopy classes of paths of a circle are in one-one correspondence with the additive group of integers [7, p.85]. It is not difficult to see that this correspondence is an isomorphism. Therefore the fundamental group of the circle is an infinite

cyclic group, that is, a group whose elements can all be expressed as powers of a single element.

Example III: The functions of the torus are infinitely many homotopy classes of paths starting and terminating at a point p of the torus shown in Figure 8,

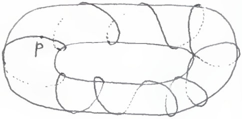

Figure 8

for if two paths of the type in Figure 8 have a different number of circuits around the torus, then they are not homotopic.

The fundamental group is an abelian group with two generators in this case; that is the group consists of elements of the form ${g_1}^a\ {g_2}^b$ where g_1, g_2 are two fixed elements such that $g_1 g_2 = g_2 g_1$, a and b being positive integers. The elements g_1 and g_2 correspond to the paths C_1 and C_2 in Figure 9 below (The meridian and parallel through p) in which the torus is represented by a rectangle with opposite sides identified.

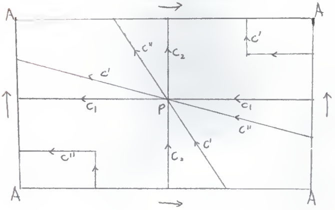

Figure 9

The fact that $g_1g_2 = g_2g_1$ corresponds to the fact that C_1C_2 is homotopic to C_2C_1. The paths C' and C'' in Figure 9 are typical members of a family of paths by means of which C_1C_2 can be deformed continuously into C_2C_1.

In the above example, the fundamental groups are abelian. However, this property is not true in general. For instance, r_1r_2 and r_2r_1 are not homotopic in $\pi_1(X,x_0)$ in Figure 10 below,

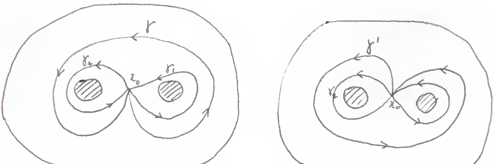

Figure 10

where $r_1r_2 \simeq r$, $r_2r_1 \simeq r'$ but r and r' are not homotopic in $\pi_1(X,x_o)$. Therefore r_1r_2 and r_2r_1 are not homotopic in $\pi_1(X,x_o)$; hence the fundamental group is abelian.

Theorem 16: Let (X,x_o) and (Y,y_o) be pairs. Then the fundamental group $\pi_1(X \times Y, x_o \times y_o)$ is isomorphic to the direct product $\pi_1(X,x_o) \otimes \pi_1(Y,y_o)$.

Proof: To show the isomorphism, it suffices to show that there exists a mapping T such that

(1) The mapping T between $\pi_1(X \times Y, x_o \times y_o)$ and $\pi_1(X,x_o) \otimes \pi_1(Y,y_o)$ is well defined,

(2) The mapping T is one-to-one,

(3) The mapping T is onto, and that

(4) The mapping T is a homomorphism.

Let P_x and P_Y denote the projections of $X \times Y$ onto X and Y respectively. If f is any mapping in $C(X \times Y, x_o \times y_o)$, the mapping P_xf and P_Yf are respectively in $C(X,x_o)$ and $C(Y,y_o)$. Let a transformation T of $\pi_1(X \times Y, x_o \times y_o)$ be defined into the direct product $P_1(X,x_o) \otimes P_1(Y,y_o)$ by $T([f]) = ([P_xf],[P_Yf])$.

First it will be shown that T is well defined. Suppose $f_o \simeq f_1$ at $x_o \times y_o$. Then there is a homotopy $G: I \times I \to X \times Y$ such that $G(x,0) = f_o(x)$, $G(x,1) = f_1(x)$, and $G(0,t) = x_o \times y_o = G(1,t)$ for $0 \le t \le 1$.

Consider the mappings P_xf_o and P_xf_1 and the mapping $P_xG: I \times I \to X$. It is clear that $P_xG(x,0) = P_xf_o(x)$, $P_xG(x,1) = P_xf_1(x)$, and $P_xG(0,t) = P_xG(1,t) = P_x(x_o \times y_o) = x_o$ for $0 \le t \le 1$.

Thus $P_X G$ is a homotopy modulo x_0 between $P_X f_0$ and $P_X f_1$, and hence the class $[P_X f]$ is well defined. A similar argument may be given to show $P_X f_1$ is well defined.

Secondly, it will be shown that T is onto. Let (g,h) be any pair in $C(X,x_0) \times C(Y,y_0)$, and let $f \in C(X \times Y,\ x_0 \times y_0)$ be defined by

$$f(x) = (g(2x), y_0), \qquad 0 \leq x \leq \tfrac{1}{2}$$

$$f(x) = (x_0,\ h(2x-1)), \qquad \tfrac{1}{2} \leq x \leq 1.$$

Then f has the property $P_X f \underset{x_0}{\simeq} g$ and $P_Y f \underset{y_0}{\simeq} h$. That f is continuous and well defined follows from the fact that $f(\tfrac{1}{2}) = (x_0, y_0)$ is uniquely defined.

Thirdly it will be shown that T is one-to-one. If $P_X f_0 \underset{y_0}{\simeq} P_X f_1$ and $P_Y f_0 \underset{y_0}{\simeq} P_Y f_1$ by the homotopies $h_1(x,t)$ and $h_2(x,t)$, let a new homotopy G be defined as

$$G\colon I \times I \to X \times Y \text{ by}$$

$$G(x,t) = (h_1(x,t),\ h_2(x,t)).$$

Then G is continuous. Also one may see that

$$G(x,0) = (P_X f_0(x),\ P_Y f_0(x)) = f_0(x),$$

$$G(x,1) = (P_X f_1(x),\ P_Y f_1(x)) = f_1(x),$$

$$G(0,t) = (h_1(0,t),\ h_2(0,t)) = (x_0, y_0),$$

$$G(1,t) = (h_1(1,t),\ h_2(1,t)) = (x_0, y_0).$$

Finally it will be shown that T is a homomorphism. If $[f]$ and $[g]$ belong to $P_1(X \times Y,\ x_0 \times y_0)$, then

$$T([f] \circ [g]) \quad T([f] * [g]) = ([P_X(f * g)], [P_Y(f * g)])$$

$$T([f] \circ [g]) \quad T([f] * [g]) = ([P_X f * P_X g], [P_Y f * P_Y g])$$

$$T([f] \circ [g]) \quad T([f] * [g]) = ([P_X f] \circ [P_X g], [P_Y f] \circ [P_Y g]).$$

This completes the proof.

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HOMOTOPY THEORY

by

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B. S., Kansas State College of Pittsburg, 1962

AN ABSTRACT OF A REPORT

submitted in partial fulfillment of the

requirements for the degree

MASTER OF SCIENCE

Department of Mathematics

KANSAS STATE UNIVERSITY
Manhattan, Kansas

1964

The theory of homotopy deals with the relations between paths defined in a topological space. Precise definitions of homotopy and allied concepts as well as theorems concerning these properties are presented.

The idea of homotopic paths leads to that of the so-called fundamental group, first introduced by the French mathematician, Poincaré. As this group is associated with homotopy theory, certain difinitions and properties of this group which are used in this theory are presented.

The essential facts from group theory as well as set theory are referred to in the bibliography.